Additional arithmetic hypotheses and polynomial definitions of the integers

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Abstract

A weak *abc* inequality with exponent 121/100 supplies the height estimate required by the thirteen-coordinate construction, together with its specified elliptic and prime-pattern premises. The ordinary discriminant–conductor form of Szpiro's conjecture implies the needed weak inequality. The truncated Vojta conjecture supplies the stronger ordinary *abc* inequality. We give the precise height comparisons and distinguish the three conjectural formulations.

1 Definitions and the required height estimates

All coordinate counts include the input. A polynomial $F \in \mathbb{Z}[X, Y_1, \dots, Y_m]$ defines $\mathbb{Z}$ in $\mathbb{Q}$ if

$$x \in \mathbb{Z} \quad \Longleftrightarrow \quad \exists \mathbf{y} \in \mathbb{Q}^m \quad F(x, \mathbf{y}) = 0.$$

For a nonzero integer n, put

$$\operatorname{rad}(n) = \prod_{p \mid n} p, \qquad \operatorname{sf}(n) = \prod_{\substack{p \mid n \\ v_p(n) \text{ odd}}} p.$$

For $t = u/v$ in lowest terms with $v > 0$, let $h(t) = \log \max(|u|, v)$. If (w, u, v) is a primitive common integer representative of $(1, a, b)$ with $w > 0$, define $h(1 : a : b) = \log \max(w, |u|, |v|)$.

Definition 1. *For $\kappa > 1$, the weak abc inequality with exponent κ is the assertion that a constant $C_\kappa > 0$ exists such that*

$$\max(|a|, |b|, |c|) \le C_\kappa \operatorname{rad}(abc)^\kappa$$

for every primitive nonzero integer triple $a + b = c$.

The ordinary rational *abc* conjecture asserts this for every $\kappa > 1$. The following result uses only one exponent.

Theorem 2. *Assume the weak abc inequality with exponent 121/100 and the elliptic group, pole obstruction and realization, and simultaneous prime-pattern inputs specified in [1]. There is a computable coefficient family in thirteen total coordinates whose sufficiently large fixed odd input shifts define $\mathbb{Z}$ in $\mathbb{Q}$. A sufficient numerical shift is not determined. The same conclusion follows under the ordinary Szpiro conjecture and the stated inherited inputs.*

Proof. We describe the numerical inequalities that make the extra hypothesis sufficient; the complete rational source and polynomial conversion are given in [1]. The rational map

$$f(t) = \frac{t(t+2)^3}{(2t+1)^3}, \qquad f(t) - 1 = \frac{(t-1)^3(t+1)}{(2t+1)^3}$$

has degree four. Its inverse images of $0, 1, \infty$ are the six rational points $0, -2, -1/2, 1, -1, \infty$. Applying the weak inequality to its numerator, denominator, and their difference gives

$$\log M(t) \geq (8/\kappa - 6)h(t) - O(1),$$

where M is the product of primes having odd contact valuation, outside a fixed exceptional set. To see the coefficient, the radical of the associated degree-six binary form is at least a constant times $\exp(4h(t)/\kappa)$, whereas $\mathrm{rad}(n)^2 \leq |n|\,\mathrm{sf}(n)$ and the form has size $O(\exp(6h(t)))$. The fixed change $t = 2 + 1/\lambda$ makes all six contacts finite.

The same weak inequality, applied to coprime eighth powers, gives

$$h(1 : \beta : T) \leq \frac{121}{437} h(\beta^8 + T^8) + O(1). \tag{1}$$

The common-denominator calculation proving (1) is included in [1]; its general coefficient is $(d/\kappa - 3)^{-1}$ for even $d > 3\kappa$. For the two arguments λ and $(\beta^8 + T^8)\lambda$, the contact bound and the local congruences therefore give

$$h(1 : \beta : T) \leq \frac{14641}{16169} \log N + O(1) \tag{2}$$

whenever $N = |\,\mathrm{num}(\beta - n)| \neq 0$. Here $\beta = 2X^2 + B$, the elliptic index n is an integer, and the one-point height theorem gives $n^2 \leq Ah(T) + J$ with fixed constants. Since $\log N \leq h(\beta) + \log(1 + |n|)$, (2) bounds both n and $h(\beta)$ independently of the fixed odd shift B. Choosing B larger than the exponential of the bound for $h(\beta)$ forces $\beta = n$. The elementary denominator calculation for $2X^2 + B$ then gives $X \in \mathbb{Z}$. Completeness and the thirteen-coordinate count follow from the explicit source in [1].

The semistable case of the ordinary Szpiro conjecture is the assertion $|\Delta_E| \leq C_\epsilon N_E^{6+\epsilon}$ for semistable elliptic curves over $\mathbb{Q}$, with minimal discriminant Δ_E and conductor N_E. Its implication of the weak *abc* inequality with every exponent $6/5 + \eta$ is established in [2, §I.3]. Take $\eta = 1/100$. □

This implication uses the discriminant–conductor conjecture itself. It does not identify that conjecture with the stronger modified Szpiro statement about $\max(|c_4|^3, |c_6|^2)$. The modified statement is equivalent to ordinary *abc*; see [2, §I.3]. The thirteen-coordinate conclusion does not remove the pole or elliptic realization hypotheses.

2 The truncated Vojta conjecture

Let X be smooth and projective over $\mathbb{Q}$, let D be a reduced divisor with normal crossings, and let A be ample. The rational-point case of the truncated Vojta conjecture asserts, outside a proper closed exceptional subset, that

$$N^{(1)}(D, P) \geq h_{K_X + D}(P) - \epsilon h_A(P) - O(1). \tag{3}$$

The counting function records a prime once when the reduction of P meets D. This is the specialization of the conjecture in [3, Conjecture 2.3].

Proposition 3. *The case of (3) for $X = \mathbb{P}^1$, $D = [0] + [1] + [\infty]$, and $A = [\infty]$ implies the ordinary rational abc conjecture.*

Proof. Write $P = a/c$ with $a + b = c$ primitive and nonzero. At finite primes the truncated counting function is $\log \mathrm{rad}(abc)$. The divisor $K_X + D$ has degree one, so its height is $h(P) + O(1)$. Thus (3) gives

$$(1 - \epsilon)h(P) \leq \log \mathrm{rad}(abc) + O(1).$$

The difference between $h(P)$ and $\log \max(|a|, |b|, |c|)$ is bounded by $\log 2$. A proper closed subset of $\mathbb{P}^1$ is finite, so its remaining rational points can be absorbed into the constant. Taking $\epsilon = \eta/(1+\eta)$ gives exponent $1+\eta$. □

Consequently this Vojta hypothesis supplies every additional *abc* input of the thirteen-coordinate and conditional quartic constructions. A lower coordinate count does not follow merely from replacing *abc* by this stronger conjecture.

3 Scope of the arithmetic hypotheses

The implications above use the stated height inequalities. Limitations of distinct prime assignments and polynomial injections, and the conditional incompatibility with Mazur's connected-component conjecture, are proved separately in [4]. Those results do not alter the arithmetic premises of Theorem 2.

References

[1] *Thirteen variables from a weak abc inequality*, companion paper and exact construction, `../conditional_few_variables/conditional13_20261109_v16.pdf`.

[2] J. Oesterlé, Nouvelles approches du *théorème* de Fermat, Séminaire Bourbaki, Exp. 694, *Astérisque* **161–162** (1988), 165–186. `https://www.numdam.org/article/SB_1987-1988__30__165_0.pdf`.

[3] P. Vojta, A more general *abc* conjecture, *Int. Math. Res. Notices* (1998), no. 21, 1103–1116. `https://arxiv.org/abs/math/9806171`.

[4] *Limitations of Auxiliary Hypotheses in Diophantine Constructions*, project manuscript, 16 November 2026, `negative_results/papers/arithmetic_hypotheses/limitations_of_auxiliary_hypotheses_20261116_v42.tex`.